The Prandtl-Meyer angle may be written[1]

$$\nu = (1/g)\tan^{-1}[g(M^2-1)^{1/2}] - \tan^{-1}(M^2-1)^{1/2} \quad (1)$$

where M is the Mach Number, $1 \leqq M \leqq \infty$,

$$g = [(\gamma-1)/(\gamma+1)]^{1/2} \quad (2)$$

and γ is the specific heat ratio. Let

$$\mu + \varepsilon = \pi/2 \quad (3)$$

where μ is the Mach angle defined by

$$\sin\mu = 1/M \quad (4)$$

From Eqs. (3) and (4)

$$(M^2-1)^{1/2} = \tan\varepsilon \quad (5)$$

Using Eq. (5), Eq. (1) may be written

$$\tan g(\nu+\varepsilon) = g\tan\varepsilon \quad (6)$$

In Eq. (6), the relation between ν and ε is still implicit. For $\gamma = \frac{5}{3}$, $g = \frac{1}{2}$ and Eq. (6) can be reduced to the explicit form

$$\tan^3(\varepsilon/2) = \tan(\nu/2) \quad (7)$$

Thus, using Eqs. (3) and (7), the simple relation may be written when $\gamma = \frac{5}{3}$

$$\mu = \pi/2 - 2\tan^{-1}[(\tan\nu/2)^{1/3}] \quad (8)$$

This analytical result was first given by Probstein[2] in terms of M, rather than μ, along with a more complicated expression when $\gamma = \frac{5}{4}$. There are no analytical expressions, $\mu = \mu(\nu, \gamma)$, for other values of γ, consequently, μ must be determined from tables, or numerically, in all other cases.

Equation (6) may be written in a form for iteration which always converges to the correct value of ε

$$\varepsilon_{i+1} = \tan^{-1}[(1/g)\tan g(\nu+\varepsilon_i)] \quad (9)$$

for $0 \leqq \varepsilon_i \leqq \pi/2$ and $0 \leqq \nu \leqq \nu_{max}$. The value ν_{max} is given by Eq. (1) for M infinite

$$\nu_{max} = (\pi/2)\{[(\gamma+1)/(\gamma-1)]^{1/2} - 1\} \quad (10)$$

One may start the iteration of Eq. (9) with an approximation for ε_1 given by Eq. (7)

$$\varepsilon_1 = 2\tan^{-1}[(\tan\nu/2)^{1/3}] \quad (11)$$

This approximation fails for large values of ν since it may give a value $\varepsilon_1 > \pi/2$. When this occurs, set $\varepsilon_1 = \pi/2$ to start the iteration with Eq. (9).

While Eq. (9) will always converge to the correct value of ε, its convergence is slow and, after two or three iterations, should be replaced by a method utilizing the derivatives. In effect, Eq. (9) is used to find an approximation of ε which lies in the range of convergence of a faster method.

One method which provides rapid convergence may be written from Eq. (6) by using a Taylor's series expansion for the separate sides about ε_i and solving for $\Delta\varepsilon$. Let

$$f_{1_i} = \tan g(\nu+\varepsilon_i) \quad (12)$$

$$f_{2_i} = g\tan\varepsilon_i \quad (13)$$

Then

$$\Delta\varepsilon_{j+1} = \left(f_{1_i} - f_{2_i} + \frac{\Delta\varepsilon_j^{\,2}}{2!}(f_{1_i}'' - f_{2_i}'') + \frac{\Delta\varepsilon_j^{\,3}}{3!}(f_{1_i}''' - f_{2_i}''') + \cdots\right) \Big/ (f_{2_i}' - f_{1_i}') \quad (14)$$

where $\Delta\varepsilon_1 = 0$ starts the iteration in Eq. (14). The advantage of this form lies in the fact that the derivatives may be written in terms of the values of f_{1_i} and f_{2_i}, i.e., f_{1_i}' and f_{2_i}' involve the $\sec^2 x$, which can be written as $1+\tan^2 x$, etc. Thus the derivatives may be included with only two numerical evaluations by the computer of the tangent functions in Eqs. (12) and (13). Depending on the number of derivatives included, two or three iterations with Eq. (14) should be sufficient. Then, of course

$$\varepsilon_{i+1} = \varepsilon_i + \Delta\varepsilon_{j+1} \quad (15)$$

is used to evaluate $f_{1_{i+1}}$ and $f_{2_{i+1}}$, etc. For $\gamma = 1.4$ and including the fourth derivative, the above procedure will provide twelve place accuracy for ε in two iterations. With the required value for ε, μ can be written using Eq. (3).

The foregoing result will be useful when studying flowfields by the method of characteristics, where at each computation point the value of μ must be determined for the resulting value of ν.

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On a Class of Fully Stressed Trusses

MOSHE FUCHS*
Technion—Israel Institute of Technology, Haifa, Israel
AND
LEWIS P. FELTON†
University of California, Los Angeles, Calif.

Introduction

PREVIOUS investigations by Dayaratnam and Patnaik[1,2] have established conditions necessary for the existence of fully stressed indeterminate trusses. It has been shown that for trusses subjected to single load conditions, only certain geometric configurations are capable of compatible displacements in a fully-stressed state. The examples presented in Refs. 1 and 2 to illustrate this fact are special cases of a general class of fully stressed trusses whose geometry is based on chords of circles. The existence of a class of structures of this type is also referred to in Ref. 3. The purpose of this Note is to describe some interesting characteristics of this class of "circle-chord" trusses which have not been previously reported.

Circle-Chord Trusses

Consider an indeterminate elastic truss composed of a single material with a specified limiting stress σ^* and subjected to a single load condition. Denote the limiting strain by ε^*, where $\varepsilon^* = \sigma^*/E$, and E = Young's modulus. The truss is defined as fully strained, and hence fully stressed, if the strain ε_i $(i = 1, \ldots, n)$ in each of the n members is a constant, ε^*. A fully stressed truss can be obtained only if there exists a set of nodal displacements compatible with the required strain field. The possibility of satisfying this condition depends solely on the configuration of the truss, and is independent of the cross-sectional areas A_i and of the external loads.[1-3] If such a set of displacements exists, and if it is also possible to specify a set of areas A_i for which internal forces equilibrate the external loads, then the desired fully stressed design is obtained. The particular class of fully stressed indeterminate trusses described herein can be made to equilibrate a wide range of loads applied at one node.

In order to establish the general nature of the trusses under investigation, consider the typical component of length l_{AB} in Fig. 1. Let δ_A and δ_B be specified magnitudes of displacements of nodes A and B, with directions given by α_{AB} and β_{AB}, respectively. It follows that component AB will be fully stressed only if its length, l_{AB}, satisfies the relation

$$l_{AB} = (\delta_A\cos\alpha_{AB} - \delta_B\cos\beta_{AB})/\varepsilon^* \quad (1)$$

Received March 7, 1974.
Index category: Optimal Structural Design.

* Graduate Student, Department of Aeronautical Engineering, A.I.Br.
† Associate Professor, Mechanics and Structures Department. Associate Fellow AIAA.

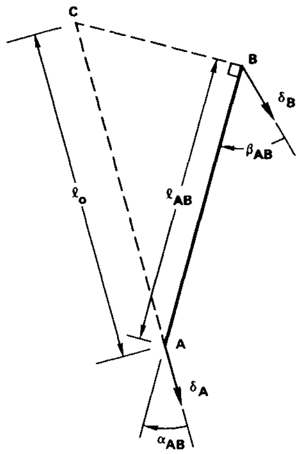

Fig. 1 Truss component.

For the special case where point B is fixed ($\delta_B = 0$), Eq. (1) may be written as

$$l_{AB} = (\delta_A/\varepsilon^*)\cos\alpha_{AB} = l_o \cos\alpha_{AB} \tag{2}$$

As may be seen from Fig. 1, the constant l_o is equal to the length of line AC, where AC is in the direction of δ_A. Physically, δ_A is equivalent to the axial deflection of a fully-stressed bar connecting A to a fixed support at C. Since ABC is a right triangle, points A, B, and C will lie on the circumference of a circle whose diameter is l_o. Furthermore, since angle α_{AB} is arbitrary, any component with length l_i, oriented at an angle α_i such that Eq. (2) is satisfied will be fully stressed, i.e., any component connecting node A to the circumference of the circle with diameter l_o will be fully stressed for the specified deflection δ_A. An assembly of these components forms an indeterminate fully stressed "circle-chord" truss, such as is shown in Fig. 2. The strains in all elements of this truss are of the same sign, as well as same magnitude. If the possibility of buckling is neglected, the concept may be extended to include a mixture of tension and compression members. Figure 3 shows one possible fully stressed configuration based on identical material properties in tension and compression. Deflection for the truss of Fig. 3 is again in the direction of the diameters. The trusses in Figs. 2 and 3 are generalizations of the symmetric and antisymmetric examples in Ref. 1.

Note that a statically determinate solution always exists as a limiting case of this class of trusses. Moreover, a determinate truss obtained from Fig. 2, in which all components lie within one circle, may be full stressed even if strains are of opposite signs in the components. (The signs of the strains will depend on the direction of the applied force.) The remainder of this discussion is restricted to the characteristics of the class of circle-chord trusses for which strains of all components within a given circle are of the same sign.

It is now necessary to establish the conditions under which the indeterminate truss in Fig. 2 can be made to equilibrate a specified force **F** applied at A. Since the strains in each element are of the same sign, the internal forces are either all tension or all compression. The equilibrium equation for node A may be written as

$$\mathbf{F} + \sum_{i=1}^{n} \mathbf{F}_i = 0 \tag{3}$$

where $\mathbf{F}_i$ is the force in component i. For fully-stressed members, Eq. (3) becomes

$$\mathbf{F} + \sigma^* \sum_{i=1}^{n} \mathbf{A}_i = 0 \tag{4}$$

The vectors $\mathbf{A}_i$ are in the directions of the elements and have magnitudes equal to the cross-sectional areas A_i. Any set of areas A_i which satisfies Eq. (4) provides a fully stressed truss with deflection δ_A. Equation (4) has an infinite number of solutions for a force **F** which lies within the sector bounded by the extreme components of the indeterminate truss, i.e., within the shaded area in Fig. 2. Figure 4 shows three possible solutions for the truss of Fig. 2.

The volume of the truss is

$$V = \sum_{i=1}^{n} A_i l_i \tag{5}$$

Substituting Eq. (2) gives

$$V = l_o \sum_{i=1}^{n} A_i \cos\alpha_i \tag{6}$$

From Eq. (4) and Fig. 2, the quantity $\sum A_i \cos\alpha_i$ in Eq. (6) may be seen to equal F_y/σ^*, where F_y is the magnitude of the y-component of **F**. Thus

$$V = F_y l_o/\sigma^* \tag{7}$$

i.e., the volume of any fully stressed truss of this class is independent of the configuration or number of components, and equals the volume of a single bar of length l_o subjected to force F_y. Note that although V is independent of F_x, the magnitude F_x is, in fact, limited by the restriction on the direction of **F**. It should be recalled that this restriction on **F** applies only to an indeterminate truss. It may also be noted that Eq. (7) also gives

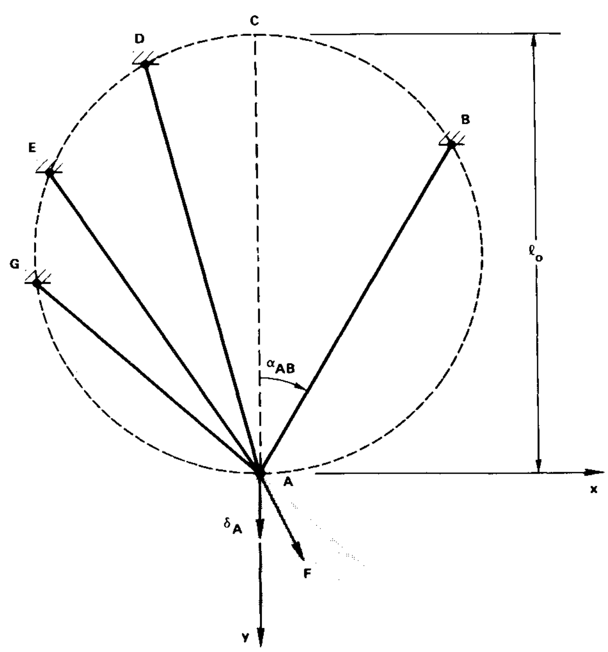

Fig. 2 Circle-chord truss.

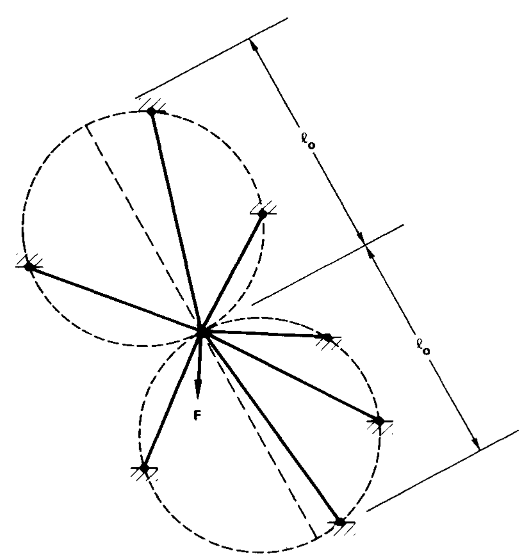

Fig. 3 Fully stressed circle-chord truss containing both tension members (top) and compression members (bottom).

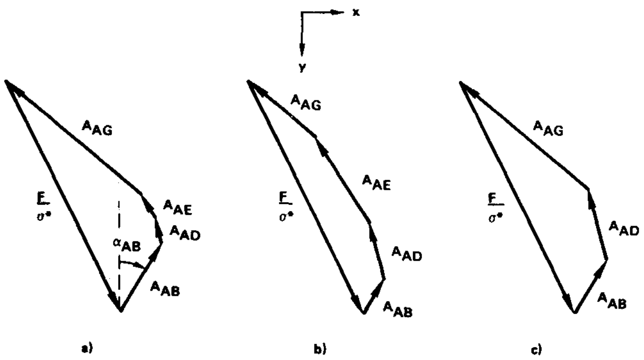

Fig. 4 Typical solutions of Eq. (4) for cross-sectional areas of truss in Fig. 2 which is subjected to force F.

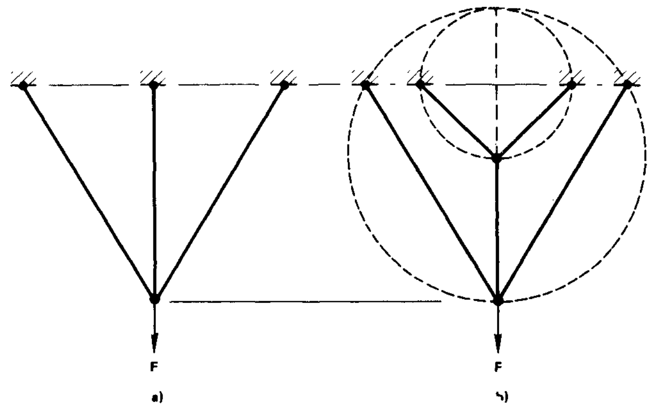

Fig. 6 a) Indeterminate truss; b) converted to fully stressed truss.

the volume of the truss in Fig. 3, since F_y may be arbitrarily apportioned between the tension and compression trusses.

Suppose, now, that the length of the component in Fig. 1 is less than the value required by Eq. (2), but that the location of point A and angle α_{AB} are unchanged. This implies that the other end point, now denoted by B', lies within the circle of diameter l_o and, from Eq. (1), that component AB' will be fully stressed for a deflection δ_A only if

$$(\delta_{B'})\cos\beta_{AB'} = \delta_A \cos\alpha_{AB} - \varepsilon^* l_{AB'} \tag{8}$$

where $\delta_{B'}$ and $\beta_{AB'}$ are values at end B' of component AB', and $l_{AB'} < l_{AB}$. Since the terms on the right-hand side of Eq. (8) are all specified, any convenient combination of $\delta_{B'}$ and $\beta_{AB'}$ which satisfies Eq. (8) may be selected, and point B' may then be treated as the free node of a second "imbedded" circle-chord truss which is to be fully stressed for the deflection $\delta_{B'}$. A resulting fully stressed truss, derived from Fig. 2, is shown in Fig. 5. Note that the direction of AB' must lie within the sector bounded by the extreme members of the second truss in order for the second truss to equilibrate force $F_{AB'}$. Also, from Eq. (2)

$$l_o' = \delta_{B'}/\varepsilon^* \tag{9}$$

Regardless of the magnitude of l_o', the circle of this diameter must intersect the original circle at point B. This may be shown as follows: assume the extension of line AB' intersects the circle of diameter l_o' at a point B''. Then, from Eqs. (2) and (8)

$$l_{B'B''} = \frac{\delta_{B'}}{\varepsilon^*}\cos\beta_{AB'} = \frac{\cos\beta_{AB'}}{\varepsilon^*}\left(\frac{\delta_A\cos\alpha_{AB} - \varepsilon^* l_{AB'}}{\cos\beta_{AB'}}\right)$$
$$= (\delta_A/\varepsilon^*)\cos\alpha_{AB} - l_{AB'} = l_{AB} - l_{AB'} = l_{B'B}$$

Therefore $B'' \equiv B$.

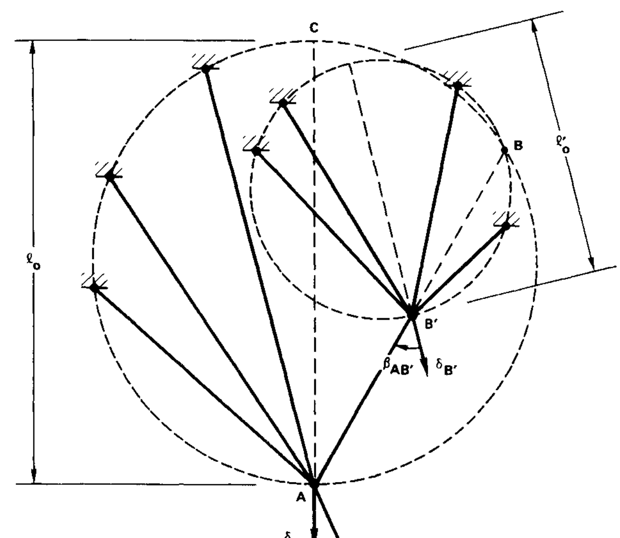

Fig. 5 Imbedded circle-chord truss.

Since $F_{AB'} = F_{AB} = \sigma^* A_{AB}$, it follows from Fig. 5, and Eqs. (7–9) and (2) that the volume of the second truss is given by

$$V' = A_{AB}(l_{AB} - l_{AB'}) \tag{10}$$

i.e., the volume of the imbedded fully stressed truss equals the volume of the missing part ($B'B$) of the primary truss.

This concept of imbedded circle-chord trusses is particularly useful in providing alternative locations for support points. For example, the truss of Fig. 6a, which cannot be fully stressed without deleting one component, may be replaced by the fully stressed design in Fig. 6b. The truss in Fig. 6b has all support points located along the same horizontal line as in Fig. 6a.

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Orthogonality Procedure for Forced Response of Multispan Beams

JOSEPH PADOVAN*
University of Akron, Akron, Ohio

Introduction

THE orthogonality property of the free vibration modes of classical beam theory are well known as evidenced by the concise treatment given by Meirovitch.[1] Because of the generality of the classical formulation of the orthogonality principle, the dynamic response of beams of constant as well as variable (EI) can be treated by the usual Sturm-Liouville (normal mode) procedure.[1] In the case of beam systems consisting of several intermediate spring supports along with possible discontinuities in (EI), the use of the classical form of orthogonality relation to develop the dynamic solution becomes awkward. With this in mind, the present Note develops a more general form of orthogonality relation. In particular, for a beam system consisting of L discrete spans with several intermediate spring-type supports, the governing beam equation for a given subspan is

Received March 18, 1974; revision received July 3, 1974.
Index category: Structural Dynamic Analysis.
* Associate Professor of Mechanical Engineering.